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FINITE GROUP EXTENSIONS OF IRRATIONAL ROTATIONS

BY

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ABSTRACT

The ergodicity of certain skew products of irrational rotations of the circle with finite groups is established with application to the construction of "well-distributed sequence generators" for finite groups.

1. Introduction

Let $X = [0,1)$ be the compact group of real numbers modulo 1, and let $0 = t_0 < t_1 < \cdots < t_r = 1$ be a partition of X. Given a group Γ on r not necessarily distinct generators $\gamma_1, \cdots, \gamma_r$, we define a function $f\colon X \to \Gamma$ by letting $f(x) = \gamma_j$ for $t_{j-1} \leqq x < t_j$, $1 \leqq j \leqq r$. If $\theta \in X$, we use θ and f to define a measurable transformation $T\colon X \times \Gamma \to X \times \Gamma$,

$$T(x,\gamma) = (x + \theta, f(x)\gamma). \tag{1.1}$$

To avoid notational complexity we have suppressed the dependence of T on θ and f. This will generally be clear from the context. Finally, if Γ is *finite*, denote the normalized Haar measure on $X \times \Gamma$ by μ.

THEOREM 1.2. *Assume*

(a) Γ *is a finite group on generators* $\gamma_1, \cdots, \gamma_r$

(b) $t_1, \cdots, t_{r-1}$ *are rational*

(c) θ *is irrational and has bounded partial quotients in its continued fraction expansion.*

Then $(T, X \times \Gamma, \mu)$ *is ergodic.*

Recall that T is ergodic if whenever $A \subseteq X \times \Gamma$ is measurable and $T^{-1}A = A$, then $\mu(A) = 0$ or 1.

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Define $f^{(n)}: X \to \Gamma$, $n \in \mathbf{Z}$, by

$$f^{(n)}(x) = \begin{cases} f(x+(n-1)\theta)f(x+(n-2)\theta)\cdots f(x) & (n>0) \\ e(=\text{identity}) & (n=0) \\ f^{-1}(x+n\theta)f^{-1}(x+(n+1)\theta)\cdots f^{-1}(x-\theta) & (n<0). \end{cases}$$

The powers of T are computed to be $T^n(x,\gamma) = (x+n\theta, f^{(n)}(x)\gamma)$. The next result says that with the hypotheses of Theorem 1.2, the sequence $f^{(n)}(x)$, $n \in \mathbf{Z}$, is "well distributed" in Γ for every $x \in X$. Below $\mathscr{X}_A$ denotes characteristic function and $|\cdot|$ cardinality.

THEOREM 1.3. *Let the assumptions be as in Theorem* 1.2. *If* $A \subseteq \Gamma$ *and* $x \in X$, *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathscr{X}_A(f^{(n+k)}(x)) = \frac{|A|}{|\Gamma|} \tag{1.4}$$

uniformly in k.

The following result will be a simple corollary to Theorems 1.2 and 1.3.

THEOREM 1.5. *With notations and assumptions as in Theorems* 1.2 *and* 1.3, *define* $n(x)$, $x \in X$, *to be the least integer* $n \geqq 1$ *such that* $f^{(n)}(x) = e$. *Then*

$$\int_0^1 n(x)dx = |\Gamma|. \tag{1.6}$$

In the next theorem notice that the hypotheses on θ and $t_1, \cdots, t_{r-1}$ are less restrictive.

THEOREM 1.7. *Let* θ *be irrational, and assume* $t_i - t_j \notin \mathbf{Z}\theta$, $0 \leqq i < j < r$. *If* Γ *is finite, there exists an integer* L *such that for any* x *the sequence* $f^{(n)}(x)$, $1 \leqq n \leqq L$, *contains every element of* Γ.

REMARK. The hypotheses of Theorem 1.7 are too weak for the conclusion of Theorem 1.2. For example, if $\Gamma = \{\pm 1\}$, $r = 2$, $\gamma_1 = 1$, $\gamma_2 = -1$, and if θ has unbounded partial quotients, there exist uncountably many t such that if $t_1 = t$, then (a) T fails to be ergodic, and (b) (1.4) fails to exist for $A = \{1\}$ and an uncountable number of x. (See [6].) However, there is no such "bad" value of t which is rational, and there remains the possibility that Theorem 1.2 is true for all irrational θ. It also may be that Theorem 1.2 is true for θ having bounded partial quotients under the assumption $t_i - t_j \notin \mathbf{Z}\theta$, $i \neq j$. This is the case for $r = 2$:

THEOREM 1.8. *With notations as in Theorem* 1.2 *assume*

(a) θ *has bounded partial quotients*

(b) $r=2$ *and* $t_1 \notin \mathbf{Z}\theta$.

Then $(T, X\times\Gamma, \mu)$ *is ergodic.*

The conclusion of Theorem 1.3 is also true with the hypotheses of Theorem 1.8. The case $r>2$ of Theorem 1.8, if true, may be rather complicated unless a proof along different lines from ours in the case $r=2$ is found.

2. A criterion for ergodicity

Let K be a compact metrizable topological group, and let $\gamma_1,\cdots,\gamma_r \in K$ be elements which generate a dense subgroup, Γ. Fix a partition $0=t_0<t_1<\cdots<t_r=1$ of X, and use it as in Section 1 to define $f\colon X\to\Gamma\subseteq K$. If $\theta\in X$ is irrational we set up $T\colon X\times K\to X\times K$ as $T(x,k)=(x+\theta, f(x)k)$. T preserves Haar measure on $X\times K$, and it makes sense to ask if T is ergodic. That it is not always so even with hypotheses (b) and (c) of Theorem 1.2, is shown by the example $X=K$, $\gamma_1=\cdots=\gamma_r=\theta$ (consider K as an additive group here). For then $T(x,k)=(x+\theta,k+\theta)$, and, for example, the set $\{(x,k)\mid x-k\in(0,\frac{1}{2})\}$ is a proper invariant set. Later in the section we will raise a question, an affirmative answer to which would imply that this is in a sense the only counterexample.

Let Λ be a complete set of continuous irreducible unitary representations of K. To each integrable function F on $X\times K$ and $\pi\in\Lambda$ we associate an operator valued function F_π on X,

$$F_\pi(x)=\int_K F(x,k)\pi(k)dk, \tag{2.1}$$

where dk is Haar measure on K. We note that if $F_\pi=0$ a.e. for every nontrivial $\pi\in\Lambda$, then for almost all x, $F(x,\cdot)$ is constant. For since Λ is countable it would be true for almost all x that $F_\pi(x)=0$, all nontrivial $\pi\in\Lambda$. For such x, $F_\pi(x,\cdot)$ is constant by the Peter-Weyl theorem.

If F is an invariant integrable function, $F(T(x,k))=F(x,k)$ a.e., then the definition of T and the fact π is a representation imply that F_π satisfies the equation

$$F_\pi(x+\theta)=\pi(f(x))F_\pi(x). \tag{2.2}$$

By the observation made in the preceding paragraph, if we are able to prove (2.2) has only the trivial solution $F_\pi=0$ for nontrivial π, then F will be constant. Now in order to prove $F_\pi=0$, we claim it is enough to prove F_π is essentially constant. For if $F_\pi = v$ a.e., then (2.2) implies $\pi(\gamma_j)v = v$, $1 \leqq j \leqq r$. Since Γ is dense in K, $\pi(k)v = v$, $k \in K$. That $v = 0$ is a consequence of the assumed nontriviality and irreducibility of π.

The next two sections are devoted to proving constancy of solutions to equations like (2.2) under the hypotheses of Theorems 1.2 and 1.8 (where Γ is finite and $\Gamma = K$). In what follows we shall describe an equation related to (2.2), whose solutions may well be constant under suitable hypotheses. If this is the case, then all solutions to (2.2) can be described.

Assume F_π is a solution to (2.2). For each $x \in X$ define a closed subgroup $L(x) \subseteq K$ by $L(x) = \{k \in K \mid \pi(k)F_\pi(x) = F_\pi(x)\}$. Using (2.2) we find

$$L(x+\theta) = f(x)L(x)f(x)^{-1}. \tag{2.3}$$

$L(\cdot)$ is measurable from X to the space of closed subsets of K with the Hausdorff topology, as it is not difficult to see.

QUESTION 2.4. Under the hypotheses of (say) Theorem 1.2, is every measurable solution to (2.3) essentially constant?

Suppose $L(\cdot)$ is a constant solution to (2.3), say $L(\cdot) = L$ a.e. Then $\gamma_j L \gamma_j^{-1} = L$, $1 \leqq j \leqq r$, and it follows that L is normal in K. In this situation it can be shown with the hypotheses of Theorem 1.2 that $\gamma_i \gamma_j^{-1} \in L$, all i, j.

Let $\mathcal{H}$ be the Hilbert space of the representation π, and let V be the linear span of the essential range of F_π in the space of linear operators on $\mathcal{H}$. By the last paragraph $\pi(\gamma_i) = \pi(\gamma_j)$ on V for all i and j. In particular, V is $\pi(K)$ invariant. Now if $h \in \mathcal{H}$, Vh is a $\pi(K)$ invariant subspace of $\mathcal{H}$. By the irreducibility of π, $Vh = \{0\}$ or $Vh = \mathcal{H}$. If it is always the first case, then $V = \{0\}$, and $F_\pi = 0$ a.e. If $Vh = \mathcal{H}$ for some h, then $\pi(\gamma_i) = \pi(\gamma_j)$, all i, j, and $\pi(K)$ is abelian. Thus $\dim(\mathcal{H}) = 1$. Identify $\mathcal{H}$ which $\mathbf{C}$, and let $\pi(\gamma_1) = \zeta \in \mathbf{C}(|\zeta| = 1)$. F_π is now complex valued, and (2.2) becomes

$$F_\pi(x+\theta) = \zeta F_\pi(x). \tag{2.2'}$$

Therefore $\zeta = \exp(2\pi i n\theta)$ for some n. Define S_n, $S_n\colon X \times X \to X \times X$, by $S_n(x, y) = (x + \theta, y + n\theta)$, and let $\arg z$, $|z| = 1$, be the value of the argument in $[0, 2\pi)$. The map $\sigma\colon X \times K \to X \times X$ defined by $\sigma(x, k) = (x, (1/2\pi)\arg \pi(k))$ satisfies $\sigma T(x, k) = S_n \sigma(x, k)$. Thus, $(S_n, X \times X)$ is a "factor" of $(T, X \times K)$. It is in this sense that an affirmative answer to 2.4 implies that the counterexample at the beginning of this section is the "only" counterexample.

REMARK. If the answer to 2.4 is "yes", and if $\gamma_i\gamma_j^{-1}$, $1 \leqq i, j \leqq r$, is also a set of generators for Γ, or at least for a dense subgroup of K, then $L = K$, and $F_\pi = 0$ a.e.

REMARK. If the answer to 2.4 is "yes" for metrizable K, it can be shown for arbitrary compact K that if hypotheses (b) and (c) of Theorem 1.2 are true, and if $\gamma_1, \cdots, \gamma_r$ generate a dense subgroup of K, then $f^{(n)}(x)$, $n \in \mathbf{Z}$, is well distributed in K for all $x \in X$. (This means (1.4) holds, with the right side replaced by the normalized Haar measure of A whenever A is an open set whose topological boundary has Haar measure 0.)

3. Proof of Theorem 1.2

Let (M, d) be a separable metric space. A measurable function $g: X \to M$ will be said to be *integrable* if for some $m \in M$ (and hence all $m \in M$) the real valued function $d(g(x), m)$ is Lebesgue integrable. Below I will be a generic letter for an interval, and $|\cdot|$ will be Lebesgue measure on X. If g is integrable, then for almost all $x \in X$

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{1}{|I|} \int_I d(g(x), g(y)) dy = 0. \tag{3.1}$$

Indeed, let $\{m_n\}$ be a dense sequence in M, and let X' be the intersection of the Lebesgue sets of the functions $d(g(x), m_n)$, $n = 1, 2, \cdots$. Then $|X'| = 1$, and if $x \in X'$, (3.1) holds. We omit the simple argument.

LEMMA 3.2. *Let $g: X \to M$ be integrable, and let $\varepsilon, \alpha > 0$. There exists $\delta > 0$ with the following property: If $\mathcal{J} = \{I\}$ is a collection of intervals with* (a) $|\bigcup_{I \in \mathcal{J}} I| > \alpha$ *and* (b) $|I| < \delta$, $I \in \mathcal{J}$, *then for some $I \in \mathcal{J}$ and $m \in M$*

$$\frac{1}{|I|} \int_I d(m, g(y)) dy < \varepsilon. \tag{3.3}$$

PROOF. Fix $\varepsilon > 0$, and let X_δ, $\delta > 0$, be the set of $x \in X$ such that if $x \in I$ and $|I| < \delta$, then

$$\frac{1}{|I|} \int_I d(g(x), g(y)) dy < \varepsilon.$$

Since $\lim_{\delta \to 0} |X_\delta| = 1$, there exists $\delta > 0$ with $|X_\delta| > 1 - \alpha$. Assumption (a) implies there exists $I \in \mathcal{J}$ such that $I \cap X_\delta \neq \varnothing$. Choose any $x \in I \cap X_\delta$, and let $m = g(x)$. Then (3.3) is true for I and m.

Fix an irrational number $\theta \in X$, and let $0 = t_0 < t_1 < \cdots < t_r = 1$ be a partition of X such that $t_i - t_j \notin \mathbf{Z}\theta$, $i \neq j$. Let Γ be a group on generators $\gamma_1, \cdots, \gamma_r$, and define $f: X \to \Gamma$ as usual. We will assume now that Γ acts on M by *isometries*. If $\gamma \in \Gamma$, $m \in M$, γm denotes the image of m under γ. We will be interested in measurable functions $g: X \to M$ which satisfy the equation

$$g(x+\theta) = f(x)g(x). \tag{3.4}$$

Special cases of (3.4) are (2.2) and (2.3). If we assume Γ has a fixed point m in M, as is true in the cases of interest, then g is automatically integrable. Indeed, $\phi(x) = d(m, g(x))$ satisfies $\phi(x+\theta) = \phi(x)$ a.e. by (2.2) and the Γ-invariance of d. Thus, ϕ is constant.

If we successively replace x by $x+\theta$, $x+2\theta$, etc. in (3.4), we find

$$g(x+n\theta) = f^{(n)}(x)g(x) \tag{3.4'}$$

(which is in fact valid for all $n \in \mathbf{Z}$). Recall that $f^{(0)}(x) = e$, and $f^{(n)}(x) = f(x+(n-1)\theta)\, f(x+(n-2)\theta) \cdots f(x)$, $n > 0$.

For each $n \geqq 0$ define a partition P_n of X by $P_0 = \varnothing$ and $P_n = \{t_i - j\theta \mid 0 \leqq j < n, 0 \leqq i < r\}$. P_n contains the discontinuities of $f^{(n)}$, and we note in particular that $t_i - n\theta \notin P_n$, $0 \leqq i < r$. Order the points of P_n as $0 = s_0^n < s_1^n < \cdots < s_{rn-1}^n < 1$. We say s_j^n is of type i, $0 \leqq i < r$, if $s_j^n = t_i - l\theta$ for some l.

Lemma 3.5. *With notations as above suppose $\varepsilon, a, b, c > o$; there exists $N < \infty$ such that the following statement is true for all $n \geqq N$ and $0 \leqq i < r$: If P_n contains at least an points s_j^n of type i such that*

$$\frac{b}{n} < |s_j^n - s_l^n| < \frac{c}{n} \qquad (l = j+1 \quad or \quad j-1) \tag{3.6}$$

(let $s_{-1}^n = s_{rn-1}^n$), there exists $v_i^n \in M$ such that

$$\frac{1}{|J|}\int_J d(v_i^n, g(y))dy < \varepsilon \quad \left(J = \left(t_i - \frac{b}{n}, t_i + \frac{b}{n}\right)\right). \tag{3.7}$$

Proof. Choose $\delta > 0$ in Lemma 3.2 for $\alpha = ab$ and $\varepsilon' = (b/c)\varepsilon$. Let $N > 2c/\delta$, and suppose $n \geqq N$ is such that P_n contains at least an points of type i such that (3.6) is true. Let $\mathcal{J}$ be the corresponding set of intervals (s_{j-1}^n, s_{j+1}^n).

The intervals in $\mathcal{I}$ may not be pairwise disjoint, but their union has measure at least $an \cdot b/n = ab$. Each $I \in \mathcal{I}$ has $|I| < 2c/n < \delta$. Therefore, there exists $I \in \mathcal{I}$, $v \in M$ such that (3.3) holds (for ε'). I has the form $(s^n_{j-1}, s^n_{j+1}) = I$ with $s^n_j = t_i - l\theta$. By (3.6) I contains the interval $L = (s^n_j - b/n, s^n_j + b/n)$. Note that $L + l\theta = J$, and $f^{(l)}$ is constant on L. Letting $v^n_i = f^{(l)}(s^n_j)v$, (3.4′), the group invariance of d, and (3.3) imply

$$\begin{aligned}\frac{1}{|J|}\int_J d(v^n_i, g(y))dy &= \frac{1}{|L|}\int_L d(v, g(y))dy \\ &\leqq \frac{c}{b}\frac{1}{|I|}\int_I d(v, g(y))dy \\ &< \frac{c}{b}\varepsilon' = \varepsilon.\end{aligned}$$

The lemma is proved.

LEMMA 3.8. *Assume $d(m,m') = 1$, $m \neq m'$, and otherwise let assumptions be as above. Given $\lambda < \infty$ suppose $\{n_k\}$ is an increasing sequence of integers such that $n_{k+1} \leqq \lambda n_k$ for all k. Suppose there exists i, $0 \leqq i < r$, and $a, b, c > 0$ such that for each k, P_{n_k} contains at least an_k points of type i such that* (3.6) *is true. There exists $v \in M$ such that*

$$\lim_{\substack{|I| \to 0 \\ t_i \in I}} \frac{1}{|I|}\int_I d(v, g(y))dy = 0. \tag{3.9}$$

PROOF. By Lemma 3.5 there exists a sequence $v^k_i = v^{n_k}_i \in M$ such that if $J_k = (t_i - b/n_k, t_i + b/n_k)$

$$\lim_{k\to\infty} \frac{1}{|J_k|}\int_{J_k} d(v^k_i, g(y))dy = 0. \tag{3.10}$$

We estimate $d(v^k_i, v^{k+1}_i)$ using the inequality $|J_k| \leqq \lambda |J_{k+1}|$:

$$\begin{aligned}d(v^k_i, v^{k+1}_i)\frac{1}{|J_{k+1}|}\int_{J_{k+1}} d(v^k_i, v^{k+1}_i)dy &\leqq \frac{1}{|J_{k+1}|}\int_{J_{k+1}} \{d(v^k_i, g(y)) + d(g(y), v^{k+1}_i)\}dy \\ &\leqq \frac{\lambda}{|J_k|}\int_{J_k} d(v^k_i, g(y))dy + \frac{1}{|J_{k+1}|}\int_{J_{k+1}} d(g(y), v^{k+1}_i)dy.\end{aligned}$$

It follows from (3.10) that $\lim_{k\to\infty} d(v_i^k, v_i^{k+1}) = 0$. By our assumption on d, $v_i^k = v_i^{k+1}$ for all large k. Let v be this common value. We will prove (3.9) holds. It is enough by symmetry to prove

$$\lim_{\substack{\beta\to 0\\ \beta>0}} \frac{1}{\beta}\int_{t_i}^{t_i+\beta} d(v, g(y))dy = 0. \tag{3.11}$$

To this end, fix $\beta > 0$ and choose n_k such that $b/n_{k+1} < \beta \leqq b/n_k$. Then

$$\frac{1}{\beta}\int_{t_i}^{t_i+\beta} d(v, g(y))dy \leqq \frac{2\lambda}{|J_k|}\int_{J_k} d(v, g(y))dy$$

and the right-hand side tends to 0 as $k = k(\beta)$ tends to ∞. The lemma is proved.

Lemma 3.12. *Assume $d(m, m') = 1$, $m \neq m'$. Suppose $\lambda < \infty$ and $\{n_k\}$ is an increasing sequence of integers such that $n_{k+1} \leqq \lambda n_k$ for all k. If there exist $b, c > 0$ such that for all k every point $s_j^{n_k}$ of P_{n_k} satisfies* (3.6), *then any measurable solution to* (3.4) *is essentially constant.*

Proof. By Lemma 3.8 there exists for each i, $0 \leqq i < r$, an element $v_i \in M$ such that (3.9) holds with $v = v_i$. Fix $\varepsilon > 0$ (to be specified later) and let $\delta > 0$ be such that if $0 \leqq i < r$, and if $t_i \in I$ and $|I| < \delta$, then

$$\frac{1}{|I|}\int_I d(v_i, g(y))dy < \varepsilon.$$

Assume k so large that $n_k > 2c/\delta$. For any j, $0 \leqq j \leqq rn_k - 1$ write $s_j^{n_k} = t_i - l\theta$ for some i and l. Define $w_j = f^{(l)}(t_i - l\theta)^{-1}v_i$. Let I_j be the interval $(s_{j-1}^{n_k}, s_{j+1}^{n_k})$. If $J = I_j + l\theta$, then because $f^{(l)}$ is constant on I_j, and because $|J| < 2c/n_k < \delta$,

$$\begin{aligned}\frac{1}{|I_j|}\int_{I_j} d(w, g(y))dy \\ = \frac{1}{|J|}\int_J d(v_i, g(y))dy \\ < \varepsilon.\end{aligned} \tag{3.13}$$

Now we estimate $d(w_j, w_{j+1})$, using the fact that

$$|I_j| \leqq \frac{2c}{b}|I_j \cap I_{j+1}|, \quad |I_{j+1}| \leqq \frac{2c}{b}||I_j \cap I_{j+1}|:$$

$$\tag{3.14} \begin{aligned} d(w_j, w_{j+1}) &= \frac{1}{|I_j \cap I_{j+1}|} \int_{I_j \cap I_{j+1}} d(w_j, w_{j+1}) dy \\ &\leqq \frac{2c}{b} \left\{ \frac{1}{|I_j|} \int_{I_j} d(w_j, y) dy + \frac{1}{|I_{j+1}|} \int_{I_{j+1}} d(y, w_{j+1}) dy \right\} \\ &< \frac{4c}{b} \varepsilon. \end{aligned}$$

Now take $\epsilon < b/4c$. We have $d(w_j, w_{j+1}) < 1$ for all j, and therefore $w_1 = w_2 = \cdots$. Denote the common value by w. By (3.13) $\{y \mid d(w, g(y)) \geqq \varepsilon^{1/2}\}$ has measure at most $2\varepsilon^{1/2}$. Letting $\varepsilon \to 0$ we see that g is essentially constant.

Proof of Theorem 1.2. Assume now that θ has bounded partial quotients. This means there is a constant $\beta > 0$ such that $\| n\theta \| > \beta / n$ for $n = 1, 2, \cdots$, where $\| \cdot \|$ is distance to nearest integer. If $t = p/q$ is rational, and if $\| t - l\theta \| < 1/q$, then it is easy to check that $\| t - l\theta \| = 1/q \cdot \| ql\theta \| > \beta / q^2 l$. From this it follows that if $t_1, \cdots, t_{r-1}$ are rational, then every interval in the partition P_n has length at least b/n for some number $b > 0$. (Let Q be a common denominator for $t_1, \cdots, t_{r-1}$. Then $b = \beta / Q^2$ will work.) It is a consequence of Lemma 4.1 and the fact θ has bounded partial quotients that there exists c such that the partition $\{-j\theta \mid 0 \leqq j < n\}$ has every interval of length at most c/n. Therefore the same is true of P_n.

We are now assuming that Γ is finite, and so we may take $K = \Gamma$ in Section 2. Let $\mathcal{H}$ be the Hilbert space of π and let $\mathcal{L}$ be the linear operators on $\mathcal{H}$. K acts on $\mathcal{L}$ through π, and the orbit space $\mathcal{L}/\pi$ is metrizable. Define $\mathcal{O}(x) \in \mathcal{L}/\pi$ by $\mathcal{O}(x) = \pi(K) F_\pi(x)$. $\mathcal{O}(\cdot)$ is measurable, and $\mathcal{O}(x + \theta) = \mathcal{O}(x)$ a.e. Therefore $\mathcal{O}(x)$ is constant, say $\mathcal{O}(x) = \mathcal{O}$ a.e. $\mathcal{O}$ has the form $\mathcal{O} = \pi(K)v$ and we let $M = \mathcal{O}$. M is a finite set, and since F_π is essentially a function from X to M, $g(x) = F_\pi(x)$ is a solution to (3.4). Placing the discrete metric on M, Lemma 3.12 applies and g is essentially constant. Theorem 1.2 is proved.

4. Proof of Theorem 1.8

The proofs of Theorems 1.2 and 1.8 are similar in spirit, but for the latter it is necessary to look more carefully at the partitions P_n.

Recall that if θ is an irrational real number with continued fraction expansion $\theta = [a_0; a_1, \cdots]$ and convergents $\{p_n/q_n\}$, then $q_{n+1} = a_{n+1}q_n + q_{n-1}$.

Below, when we say $x \in X$ is to the right (resp. left) of 0, we shall mean $x \in (0, \frac{1}{2})$ (resp. $x \in (\frac{1}{2}, 1)$). Then x is to the right (left) of y if $x - y$ is to the right (left) of 0.

The next lemma is probably well known, but we know of no reference for it.

LEMMA 4.1. *Let $\theta = [a_0, a_1, \cdots]$ and $\{p_n/q_n\}$ be as above. Given a positive integer N define n and α, $0 \leqq \alpha < a_{n+1}$, by $q_n < N \leqq q_{n+1}$ and $q_{n-1} + \alpha q_n < N \leqq q_{n-1} + (\alpha + 1)q_n$. If $q_n\theta$ is to the right (left) of 0, and if $0 \leqq l < N$, then among the numbers $j\theta, 0 \leqq j < N, j \neq l$, the one which is closest to $l\theta$ on its right (left) is the one with j given by*

$$
\begin{aligned}
&j = l + q_n && (0 \leqq l < N - q_n)\\
&j = l - (q_{n-1} + (\alpha - 1)q_n) && (N - q_n \leqq l < q_{n-1} + \alpha q_n) \qquad (4.2)\\
&j = l - (q_{n-1} + \alpha q_n) && (q_{n-1} + \alpha q_n \leqq l < N).
\end{aligned}
$$

PROOF. Since $0 \leqq l, j < N \leqq q_{n+1}$, if $l \neq j$ $\|(l - j)\theta\| \geqq \|q_n\theta\|$, and equality can hold only if $l - j = \pm q_n$. Therefore, if $0 \leqq l < N - q_n$, $j = l + q_n$, as claimed in the first line of (4.2).

For any integer m the numbers $(q_{m-1} + \beta q_m)\theta$, $0 \leqq \beta < a_{m+1}$, are $\|q_n\theta\|$ dense on the short interval between 0 and $q_{m-1}\theta$. If $j\theta$ lies in this interval, then for some β, $\|(j - (q_{m-1} + \beta q_m))\theta\| < \|q_m\theta\|$. If $0 \leqq j < q_{m+1}$, then necessarily $j = q_{m-1} + \beta q_m$.

Assume $q_n\theta$ is to the right of 0. A similar argument works for the left. If $N - q_n \leqq l < N$, and if $l < j < N$ is such that $j\theta$ is a distance less than $\|q_{n-2}\theta\|$ from $l\theta$ on its right, then by the preceding paragraph $j - l = q_{n-2} + \beta q_{n-1}$. If $\beta < a_n - 1$, then $\|(j - l)\theta\| > \|q_n\theta\| + \|q_{n-1}\theta\|$. Whichever of the second two inequalities in (4.2) l satisfies, the value of j associated to l satisfies $\|(j - l)\theta\| \leqq \|q_n\theta\| + \|q_{n-1}\theta\|$, with $j\theta$ on the right of $l\theta$ and strict inequality unless $\alpha = 0$ and $N - q_n \leqq l < q_{n-1}$. We conclude that if $N - q_n \leqq l < j < N$, and if $j\theta$ is the nearest neighbor to $l\theta$ on the right, then $\alpha = 0$ and $j = l - (q_{n-1} - q_n)$ $(= l + q_{n-2} + (a_n - 1)q_{n-1})$.

If $0 \leqq j < l < N$, and if $j\theta$ is a distance at most $\|q_{n-1}\theta\|$ from $l\theta$ on its right, then by the paragraph preceding the last, $j = l - (q_{n-1} + \beta q_n)$. Moreover, the smallest possible distance is attained using the maximum β such that $l - (q_{n-1} + \beta q_n) \geqq 0$. In the interval $N - q_n \leqq l < N$, this value is $\beta = \alpha$ or $\alpha - 1$ according to (4.2), assuming $\alpha \geqq 1$ in the $\alpha - 1$ case. The remaining case is

$\alpha = 0$, $N - q_n \leqq l < q_{n-1}$. Here there is no value j, $0 \leqq j < l$, such that $\|(j-l)\theta\| \leqq \|q_{n-1}\theta\|$. If $0 \leqq j < N$, if $j\theta$ is on the right of $l\theta$, and if $\|(j-l)\theta\| < 2\|q_{n-1}\theta\|$, then $\|(j-l)\theta - (l-(q_{n-1}-q_n)-l)\theta\| < \|q_{n-1}\theta\|$. Since $|j-l+q_{n-1}-q_n| < q_n$ if $j \geqq 0$ and $0 \leqq l < q_{n-1}$, $j = l-(q_{n-1}-q_n)$. This completes the proof of the lemma.

Remark. Assume θ has bounded partial quotients. Let $\|l\theta\| \geqq c/l, l \geqq 1$. The largest possible value which can occur in (4.2) is

$$\|q_n\theta\| + \|q_{n-1}\theta\| \leqq 2\|q_{n-1}\theta\| < \frac{2}{q_n} \leqq \frac{2}{c}\,\frac{1}{q_{n+1}} \leqq \frac{2}{c}\,\frac{1}{N}.$$

(If $\|l\theta\| \geqq c/l$, $l \geqq 1$, then $c/q_n \leqq \|q_n\theta\| < 1/q_{n+1}$ implies $1/q_n < 1/c \cdot 1/q_{n+1}$.) This fact was needed in the proof of Theorem 1.2.

In what follows we assume θ has bounded partial quotients. Thus, there exists $c > 0$ such that $\|q\theta\| \geqq c/q$ for all $q \geqq 1$. Fix $t \in X$ such that $t \not\in Z\theta$, set $r = 2$, and define $t_0 = 0$, $t_1 = t$, $t_2 = 1$. Let Γ be a group on generators γ_1, γ_2, define $f: X \to \Gamma$ as usual, and assume (M,d) is a separable metric space on which Γ acts by isometries. Finally, fix a measurable function $g: X \to M$ such that

$$g(x+\theta) = f(x)g(x). \tag{4.3}$$

We will prove g is essentially constant if $d(m,m') = 1$, $m \neq m'$, and as in Section 3, Theorem 1.8 follows.

Define $N_n = q_n + q_{n-1}$. Since $q_{n+1} \leqq 1/c \cdot q_n$ $(c/q_n \leqq \|q_n\theta\| < 1/q_{n+1})$, $N_{n+1} \leqq 1/c \cdot N_n$. Suppose for some $\delta > 0$ and all n every interval of the partition P_{N_n} (Section 3) has length at least $\delta\|q_n\theta\| > c\delta/q_n > c\delta/N_n$. Then by Lemma 3.12, g is essentially constant. Therefore, we will assume in what follows that no such δ exists.

Let $\varepsilon_n = \min_{|s|<N_n}\|t+s\theta\|$. The smallest interval in P_{N_n} has length $\min(\varepsilon_n, \|q_n\theta\|)$ by Lemma 4.1. Therefore, if the smallest length is less than $\delta\|q_n\theta\|$, $\delta < 1$, it must be ε_n.

Given δ, $0 < \delta < 1$, let n_0 be the least integer such that $P_{N_{n_0}}$ has an interval of length less than $\delta\|q_{n_0}\theta\|$. Note that $n_0 = n_0(\delta) \to \infty$ as $\delta \to 0$. Let s, $|s| < N_{n_0}$, be such that $\varepsilon_{n_0} = \|t+s\theta\|$. We treat the case $s \geqq 0$, the case $s < 0$ being similar. By our choice of n_0, s must satisfy the inequality $N_{n_0-1} \leqq s < N_{n_0}$, or more simply, $q_{n_0-1} < s < q_{n_0+1}$. Let $n = n_0 + 5$. If δ is sufficiently small, every interval in P_{N_n} has length at least $\|s\theta + t\|$. For if $\|s'\theta + t\| \leqq \|s\theta - t\|$, then $\|(s-s')\theta\| \leqq 2\delta\|q_{n-5}\theta\| \leqq 2\delta(1/c)^7\|q_{n+2}\theta\|$. Since $|s'| < q_n + q_{n-1} \leqq q_{n+1}$, $\|s - s'| < q_{n+2}$. If $\delta < \frac{1}{2}c^7$, then $s = s'$.

We fix n as in the previous paragraph (assuming $\delta < \frac{1}{2}c^7$). Note that s satisfies the inequality $q_{n-6} < s < q_{n-4}$.

Now $q_n\theta$ is either to the right or to the left of 0. Considerations of symmetry make it sufficient to treat the former case. Thus, we assume below $q_n\theta$ is to the right of 0.

Fix u, $0 \leqq u < q_{n-1}$. We wish to list a string of successive elements $-l\theta$ from left to right, $l < N_n$, beginning with $-(u+q_n)\theta$ and ending with the first point $-v\theta$ to the right of $-u\theta$ such that $0 \leqq v < q_{n-1}$. To do this we apply Lemma 4.1, noting that when $N_n = q_n + q_{n-1}$, $\alpha = 0$ in (4.2), and only the alternatives $(l+q_n)\theta$, $(l-q_{n-1})\theta$ are possible there. Of course, since we are working with negative numbers, the successors to $-l\theta$ on the right have possible forms $-(l-q_n)\theta$, $-(l+q_{n-1})\theta$. The lists (A) and (B) correspond to $0 \leqq u < q_{n-2}$ and $q_{n-2} \leqq u < q_{n-1}$ respectively. We delete θ from $-j\theta$:

(A) $-(u+q_n), -u, -(u+q_{n-1}), \cdots, -(u+(a_n+1)q_{n-1}),$
$$-(u+(a_n+1)q_{n-1}-q_n)$$

(B) $-(u+q_n), -u, -(u+q_{n-1}), \cdots, -(u+a_nq_{n-1}), -(u+a_nq_{n-1}-q_n).$

LEMMA 4.3. *Let $0 \leqq u < q_{n-1}$, and let I be the short interval between the first and last points of whichever of strings* (A) *and* (B) *applies to u. If $\delta > 0$ is sufficiently small, then $f^{(p)}$ is constant on I, where p is given by*

i. $p = u - s$ *if* $0 \leqq u - s < q_{n-2}$

ii. $p = u - s + a_nq_{n-1} - q_n$ *if* $q_{n-2} \leqq u - s < q_{n-1}$

iii. $p = u$ *if* $u < s$.

The proof of the lemma will be given at the end of this section. If I is in class (A), meaning $0 \leqq u < q_{n-2}$, then only i and iii above are possible. For class (B) it is only i and ii, because $u \geqq q_{n-2} > q_{n-4} > s$ implies iii is impossible. The possibilities (A–i), (B–i), etc., will be referred to as the *type* of I.

Let u, I, and p be as in Lemma 4.3. Then $-u+p$ does not depend upon u. Since I is determined by (A) or (B), the interval $I + p\theta$ depends only upon the type of I. This is significant because, as we shall see, there is a constant $\alpha > 0$ such that for each of the four possible types the totality of intervals of that type has measure at least α. Assume this to be so for the moment. Given $\varepsilon > 0$, $\delta > 0$ can be chosen so that $n = n(\delta)$ is large as we please. Therefore, by Lemma 3.2 if δ is small, there will exist for any type some interval I of that type such that (3.3) holds; that is, for some $m \in M$

$$(4.4) \qquad \frac{1}{|I|}\int_I d(m, g(y))dy < \varepsilon.$$

Let $J = I + p\theta$, $m' = f^{(p)}(x)m$ for any $x \in I$. Then by (3.4), (4.4), and the group invariance of d

$$(4.5) \qquad \frac{1}{|J|}\int_J d(m', g(y))dy < \varepsilon.$$

Now if I' is any other interval of the type of I, and if p' is associated to I', define $m'' = f^{(p')}(x)^{-1}m'$, for any $x \in I'$. Then (4.5) and (3.4) imply

$$\frac{1}{|I'|}\int_{I'} d(m'', g(y))dy < \varepsilon.$$

In other words, g is nearly constant, in the sense of (3.3), on every interval of every type if δ is sufficiently small.

Let I, I' be successive intervals (of possibly different types). Then $I \cap I'$ is an interval of length $\| q_n\theta \|$. Since I and I' have lengths bounded by

$$2\| q_n\theta \| + (a_n + 1)\| q_{n-1}\theta \| = \| q_n\theta \| + \| q_{n-1}\theta \| + \| q_{n-2}\theta \|$$
$$\leqq \left(1 + \frac{1}{c} + \frac{1}{c^2}\right)\| q_n\theta \|,$$

an argument as in (3.14) shows that if m, m' are associated to I and I' as in (4.4), then $d(m, m') \leqq 2(1 + c^{-1} + c^{-2})\varepsilon$. If ε is small, and if $d(m, m') = 1$, $m \neq m'$, it follows that every interval of every type has the same constant associated to it. Letting $\varepsilon \to 0$, we see that g is essentially constant.

Let us now estimate the occurrences of types (A–i), (B–i), etc.

Type (A–i) occurs when u satisfies the inequalities $0 \leqq u < q_{n-2}$ and $0 \leqq u - s < q_{n-2}$. Since $s < q_{n-4}$, there are at least $q_{n-2} - q_{n-4} \geqq q_{n-3}$ such u. Every interval of every type contains at least one segment of length $\| q_{n-1}\theta \|$ which intersects no other interval (A) or (B). Therefore the totality of intervals of type (A–i) measures at least $q_{n-3}\| q_{n-1} \| > c^3$.

Type (B–i) occurs when u satisfies $q_{n-2} \leqq u < q_{n-1}$ and $0 \leqq u - s < q_{n-2}$. Since $s > q_{n-6}$, there are at least q_{n-6} such u. The type (B–i) intervals have total measure at least $q_{n-6}\| q_{n-1}\theta \| > c^6$.

Type (B–ii) occurs when $q_{n-2} \leqq u < q_{n-1}$ and $u - s \geqq q_{n-2}$. Since $q_{n-2} + s < q_{n-2} + q_{n-4} < q_{n-3}$, this happens at least $q_{n-1} - q_{n-3} \geqq q_{n-2}$ times. A lower bound for the total measure of intervals of this type is $q_{n-2}\| q_{n-1}\theta \| \geqq c^2$.

Finally, type (A–iii) occurs when $0 \leqq u < q_{n-2}$ and $u < s$. Since $s > q_{n-6}$, there are at least q_{n-6} such u. The total measure in this case is greater than c^6.

The proof of Theorem 1.8 is complete, except for the proof of Lemma 4.3.

PROOF OF LEMMA 4.3. Fix $0 \leqq u < q_{n-1}$, and let I be the interval corresponding to the string, (A) or (B), associated to u. We will compute the possible values of j such that $t - j\theta \in I$. Then, letting p be the smallest integer such that $-p\theta \in I$ or $t - p\theta \in I$, the function $f^{(p)}$ has no discontinuities on I. The lemma will be a consequence therefore of our explicit computation.

Let $-l\theta$, $-l'\theta$ be successive elements of I. We will examine the possibilities for points $t - j\theta$ between the two.

If $l, l' \geqq s$, then one of the points $t-(l-s)\theta$, $t-(l'-s)\theta$ will be between the two (remember that $\|s\theta + t\| < \delta \| q_n\theta \|$, and $\| q_n\theta \|$ is the length of the shortest of the segments comprising I). Obviously, $t-(l-s)\theta$, $t-(l'-s)\theta$ are successive elements in the partition $\{t - j\theta\}$. (For $t - j\theta$ to lie between the two it is necessary, by Lemma 4.1, that either $j = l - s - q_n$, in which case $l > q_n$ and $l' = l - q_n$, meaning $j = l' - s$; or that $l' - s = j - q_n$, meaning that $l' < q_{n-1} + s$ and hence $l' = l + q_{n-1}$ is impossible. Thus, $l' = l - q_n$, and $j = l' - s + q_n = l - s$). Thus, if $t - j\theta$ is to be a second point of its kind between $-l\theta$ and $-l'\theta$, it is necessary that it be between $-l\theta$ and $t-(l-s)\theta$ or between $-l'\theta$ and $t-(l'-s)\theta$. In either case, $\varepsilon_n < \| s\theta + t \|$, contradicting the definition of s.

Now consider the case $l' < s$. Since $s < q_{n-1}$, $l = l' + q_n$. If δ is sufficiently small, the only possibility for j such that $t - j\theta$ is between $-l\theta$ and $-l'\theta$ is $j = l - s$. For in the partition $\{t - i\theta\}$ the successor to $t-(l-s)\theta$ on the right is, since $l - s < q_n$, $t-(l-s+q_{n-1})\theta$. This point cannot lie between $-l\theta$ and $-l'\theta$ so long as $\| q_{n-1}\theta \| > (1+\delta)\| q_n\theta \| > \| q_n\theta \| + \| s\theta + t \|$. (The latter is true when $\delta < c$ because $\| q_{n-1}\theta \| - \| q_n\theta \| \geqq \| q_{n+1}\theta \| \geqq c/q_{n+1} > c\| q_n\theta \|$). Any other point $t - j\theta$ between $-l\theta$ and $-l'\theta$ would lead to a smaller value of ε_n as above.

Finally, suppose $l < s$. Here the only possibility is $l = u$ because all other points of I, save the one on the far right, have $l \geqq q_{n-1}$. Here we see there are two possibilities for points between $-u\theta$ and $-(u + q_{n-1})\theta$. Namely, $t - (u + q_n - s + q_{n-1})\theta$ and $t-(u - s + q_{n-1})\theta$. Notice that $u + q_n - s + q_{n-1} > u$ and $u - s + q_{n-1} > u$.

Collecting results we have found there is only one possibility for $t - j\theta \in I$ in which j is not $l - s$ where $-l\theta \in I$. This occurs when $u < s$ and is $j = u + q_n - s + q_{n-1} > u$. It is now an easy matter to compute the minimum value of p such that $-p\theta \in I$ or $t - p\theta \in I$. If $u \geqq s$, it is $p = u - s$ unless $u \geqq s + q_{n-2}$. In the latter case we are in case (B) and $p = u - s + a_n q_{n-1} - q_n < u - s$. Finally, if $u < s$, we are in case (A), and the only possible competitor is $u + (a_n + 1)q_{n-1} - q_n - s$. Since $(a_n + 1)q_{n-1} - q_n - s = q_{n-1} - q_{n-2} - s \geqq q_{n-3} - s > 0$, u is the minimum value. This completes the proof of Lemma 4.3.

5. Proof of Theorems 1.3 and 1.5

Let Γ be a group on generators $\gamma_1, \cdots, \gamma_r$, let $\theta \in X$ be irrational, and let t_0, $t_1, \cdots, t_r$ be a partition as in Section 1. We define $f: X \to \Gamma$ as usual. If π is a finite dimensional unitary representation of Γ, let $\mathcal{H}_\pi$ be the Hilbert space of π. By Lemma 1′ of [5], if there exists $x \in X$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi(f^{(n+k)}(x)) = 0 \tag{5.1}$$

fails to hold uniformly in k, then there exists a nontrivial measurable function F_π, $F_\pi: X \to \mathcal{H}_\pi$, which satisfies the equation

$$F_\pi(x + \theta) = f(x)F_\pi(x). \tag{5.2}$$

Under the hypotheses of Theorem 1.3 (or Theorem 1.8), F_π must be essentially constant. Thus, if π is irreducible and nontrivial, $F_\pi = 0$, and (5.2) has no nontrivial solution. That is, (5.1) holds uniformly in k for all x. By Weyl's criterion (Γ is now finite), if $A \subseteq \Gamma$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{X}_A(f^{(n+k)}(x)) = \frac{|A|}{|\Gamma|} \tag{5.3}$$

uniformly in k, and Theorem 1.3 is proved.

We now prove Theorem 1.5. Let $\Omega = X \times \{e\}$, and let ν be the normalization of $\mu|_\Omega$ (μ = Haar measure on $X \times \Gamma$). The "induced transformation" of Kakutani [2] is defined as follows: $S(x,e)$ is the first point among $T(x,e)$, $T^2(x,e), \cdots$ which belongs to Ω. Clearly $S(x,e) = (x + n(x)\theta, e)$. (S, Ω, ν) is measure preserving and ergodic. Let $M > 0$ and let $N = n(x) + n(Sx) + \cdots + n(S^{M-1}x)$. Then if $A = \{e\}$,

$$\frac{1}{N} \sum_{n=1}^{N} \mathcal{X}_A(f^{(n)}(x)) = \frac{M}{N}$$

and therefore, as $M \to \infty$, $M/N \to 1/|\Gamma|$. On the other hand, by the ergodic theorem, as $M \to \infty$

$$\frac{N}{M} \to \int_0^1 n(y)dy$$

for almost all x. We conclude that

$$\int_0^1 n(y)dy = |\Gamma|$$

and Theorem 1.5 is proved.

6. Proof of Theorem 1.7

Parts of the proof of Theorem 1.7 will only be sketched because the techniques are so similar to those employed in [6]. Our primary purpose in including any proof has been to point out the further significance of (2.3) which may well be the key to the case of compact K (infinite Γ) in Theorems 1.2 and 1.8.

Let K, Γ, f, θ, etc. be as in the first paragraph of Section 2. In addition we assume $t_i - t_j \notin \mathbf{Z}\theta$, $i \neq j$.

For the purposes of Theorem 1.7 it is no loss of generality to suppose f has no nonzero periods. For let $X_0 \subseteq X$ be the group of periods of f. X_0 is closed because f is right continuous. If $X_0 = X$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_r$, and Γ is cyclic. The conclusion of Theorem 1.7 is trivial in this case. If X_0 is finite, then $X/X_0 \approx X$, and by redefining f, θ, t_i, etc. in terms of the quotient, we obtain a nonperiodic f. If the conclusion of Theorem 1.7 is true in the quotient situation, it is true in the original situation.

Let $Y = \{\gamma_1, \cdots, \gamma_r\}^{\mathbf{Z}}$, and let $\sigma: Y \to Y$ be the left shift: $(\sigma y)_n = y_{n+1}$, $y = \{y_n\}$. To each point $x \in X$ we associate a point $w_x \in Y$, defining $w_x(n) = f(x + n\theta)$. Note that because $w_{x+\theta} = \sigma w_x$, $W = \text{closure } \{w_x \mid x \in X\}$ is σ-invariant. Because f has no nonzero period, it can be shown that (a) $\tau w_x = x$ is well defined on $\{w_x \mid x \in X\}$, and (b) τ extends to be a continuous map from W to X. Let $P = \{t_i - j\theta \mid 0 \leqq i \leqq r-1, j \in \mathbf{Z}\}$. For any $x \in X$ there is at most one value of n such that $x + n\theta \in \{t_0, \cdots, t_{r-1}\}$ (because $t_i - t_j \notin \mathbf{Z}\theta$, $i \neq j$), and such a value exists if and only if $x \in P$. We list some easily verified properties of τ (see [6] for the techniques):

(c) $\tau w = x$ implies $w(n) = w_x(n)$ whenever $x + n\theta \notin \{t_0, \cdots, t_{r-1}\}$.

(d) If $w \neq w_x$, and if $\tau w = x$, then $w(n) = \gamma_{i-1}$ if $x + n\theta = t_i$ ($= \gamma_r$ if $x + n\theta = 0$).

Property (c) implies $\tau^{-1}x$ is a singleton ($= \{w_x\}$) for $x \notin P$. Property (d) implies $\tau^{-1}x$ has at most two points for $x \in P$. For some i $\gamma_{i-1} \neq \gamma_i$ because f has no periods, and if $x + n\theta = t_i$, $\tau^{-1}x$ will have two points.

LEMMA 6.1. *If $w_1, w_2 \in W$, and if $\tau w_1 = \tau w_2$, then either $w_1(n) = w_2(n)$, $n \geqq 0$, or else $w_1(n) = w_2(n)$, $n < 0$.*

PROOF. Immediate from (c) above.

LEMMA 6.2. *For all $w \in W \{\sigma^n w \mid n \in \mathbf{Z}\}$ is dense in W. ((σ, W) is a minimal flow.)*

PROOF. Differs in no essential respect from the proof on page 4 of [6].

Let $Z = X \times K$, and define $T\colon Z \to Z$ by $T(w,k) = (\sigma w, w(0)k)$. Define $z_n(\cdot)$ on W, $n \in Z$, by

$$z_n(w) = \begin{cases} w(n-1)w(n-2)\cdots w(0) & (n>0) \\ e & (n=0) \\ w(n)^{-1}w(n+1)^{-1}\cdots w(-1)^{-1} & (n<0). \end{cases} \tag{6.3}$$

The powers of T are computed to be

$$T^n(w,k) = (\sigma^n w, z_n(w)k). \tag{6.4}$$

(Compare with Section 1.)

For each $w \in W$ define $\Lambda(w) \subseteq K$, to be the set of k such that $(w,k) \in \mathcal{O}((w,e))$, where $\mathcal{O}((w',k'))$ denotes the orbit closure of $(w',k') \in Z$ under T. To say $k \in \Lambda(w)$, is to say there exists a sequence $\{n_j\}$ such that $z_{n_j}(w) \to k$ and $\sigma^{n_j} w \to w$. This together with (6.3) make it evident that if $k \in K$ is arbitrary

$$\{k' \in K \mid (w,k') \in \mathcal{O}((w,k))\} = \Lambda(w)k. \tag{6.5}$$

In particular, $\Lambda(w)\Lambda(w) \subseteq \Lambda(w)$, and $\Lambda(w)$ is a closed subsemigroup of K. Thus, $\Lambda(w)$ is a closed *subgroup* of K.

For all $(w,k) \in Z$ the flow $(T, \mathcal{O}((w,k)))$ is minimal. Indeed, if $(w',k') \in \mathcal{O}((w,k))$, there is a sequence $\{n_j\}$ such that $T^{n_j}(w',k') \to (w,k'')$ because (σ, W) is minimal and K is compact metric. By (6.3), $k'' = \lambda k$ for some $\lambda \in \Lambda(w)$. Since $\lambda^{-1} \in \Lambda(w)$, $(w,k) \in \mathcal{O}((w,\lambda k))$. Thus, $(w,k) \in \mathcal{O}((w',k'))$, and so $\mathcal{O}((w,k)) \subseteq \mathcal{O}((w',k')) \subseteq \mathcal{O}((w,k))$. That is, $\mathcal{O}((w',k')) = \mathcal{O}((w,k))$, and $(T, \mathcal{O}((w,k)))$ is minimal.

LEMMA. 6.6. *For all $n \in \mathbf{Z}$ and $w \in W$*

$$\Lambda(\sigma^n w) = z_n(w)\Lambda(w)z_n(w)^{-1}. \tag{6.7}$$

PROOF. By (6.5) $\Lambda(\sigma^n w)z_n(w) = \{k \mid (\sigma^n w, k) \in \mathcal{O}((\sigma^n w, z_n(w)))$. Apply T^{-n} and the fact $z_{-n}(\sigma^n w) = z_n(w)^{-1}$ to conclude $z_n(w)^{-1}\Lambda(\sigma^n w)z_n(w) \subseteq \Lambda(w)$. Now if $\lambda \in \Lambda(w)$, there exists by the minimality of $(T, \mathcal{O}((w,e)))$ a sequence $\{n_j\}$ such that $T^{n_j}(\sigma^n w, z_n(w)) \to (w, \lambda)$. Then $T^{n_j+n}(\sigma^n w, z_n(w)) \to (\sigma^n w, \lambda' z_n(w))$, $\lambda' \in \Lambda(\sigma^n w)$, and so $\lambda = z_n(w)^{-1}\lambda' z_n(w)$. Thus, (6.7) holds.

In what follows we let $\mathcal{P}_c(K)$ be the space of closed subsets of K, and we regard $\Lambda(\cdot)$ as a function, $\Lambda\colon W \to \mathcal{P}_c(K)$.

LEMMA 6.8. *With notations as above, $\Lambda(\cdot)$ is a continuous function.*

PROOF. Fix $w \in W$, and suppose $(w', k) \in \mathcal{O}((w, e))$. We will prove $\Lambda(w') = k\Lambda(w)k^{-1}$ (compare with (6.7)). Assuming this to be so, it is clear from the definitions that if $w'_n \to w$, if $(w'_n, k'_n) \in \mathcal{O}((w, e))$, and if $k'_n \to k'$, then $k' \in \Lambda(w)$. This implies $\Lambda(w') \to \Lambda(w)$ as $w' \to w$.

To prove $\Lambda(w') = k\Lambda(w)k^{-1}$, first use (6.5) as in Lemma 6.6 to prove $k^{-1}\Lambda(w')k \subseteq \Lambda(w)$. If $\lambda \in \Lambda(w)$, choose $\{n_j\}$ such that $T^{n_j}(w', k) \to (w, \lambda)$ using minimality. Then $\lambda = \beta k$, where $z_{n_j}(w') \to \beta$. Now clearly $k\beta \in \Lambda(w')$, and therefore $\lambda = k^{-1}(k\beta)k \in k^{-1}\Lambda(w')k$. The lemma is proved.

LEMMA 6.9. *With notations as above, if $\tau w = \tau w'$, then $\Lambda(w) = \Lambda(w')$.*

PROOF. By Lemma 6.1 either $w(n) = w'(n)$, $n \geqq 0$, or else $w(n) = w'(n)$, $n < 0$. This implies either $z_n(w) = z_n(w')$, $n \geqq 0$, or else $z_n(w) = z_n(w')$, $n < 0$. Whichever of the two cases applies, denote the common value by z_n. We have by (6.7)

$$\begin{aligned} \Lambda(\sigma^n w) &= z_n \Lambda(w) z_n^{-1} \\ \Lambda(\sigma^n w') &= z_n \Lambda(w') z_n^{-1} \end{aligned} \tag{6.10}$$

either for all $n \geqq 0$ or else for all $n < 0$. Let $d(\cdot,\cdot)$ be any compatible metric on W. Then by property (c) at the beginning of this section, $\lim_{n\to\infty} d(\sigma^n w, \sigma^n w') = 0$. Letting δ be the Hausdorff metric on $\mathcal{P}_c(K)$, the continuity of $\Lambda(\cdot)$ and the compactness of W imply $\lim_{n\to\infty} \delta(\Lambda(\sigma^n w), \Lambda(\sigma^n w')) = 0$. Then by (6.10) and the compactness of K, there exists $z \in K$ such that $z\Lambda(w)z^{-1} = z\Lambda(w')z^{-1}$. Thus $\Lambda(w) = \Lambda(w')$. The lemma is proved.

If $x \in X$, define $\Lambda(x) = \Lambda(w_x)$. By Lemma (6.9) and the continuity of $\Lambda(\cdot)$ on W, Λ is continuous on X. Since $\tau\sigma w = \tau w + \theta$, the function on X satisfies the equation

$$\Lambda(x + \theta) = f(x)\Lambda(x)f(x)^{-1}. \tag{6.11}$$

QUESTION 6.12. Let θ, Γ, etc. be as above. Is every continuous solution to (6.11) necessarily constant?

REMARK. Notice that the continuity of Λ implies

$$\gamma_i \Lambda(t_i)\gamma_i^{-1} = \gamma_{i+1}\Lambda(t_i)\gamma_{i+1}^{-1},\ 1 \leqq i < r, \text{ and } \gamma_r\Lambda(0)\gamma_r^{-1} = \gamma_1\Lambda(0)\gamma_1^{-1}.$$

There are two obvious instances in which the answer to (6.12) is "yes". If K is *abelian*, then (6.11) implies $\Lambda(x + \theta) = \Lambda(x)$, all x, and constancy follows from Kronecker's theorem and continuity. Secondly, if K is finite ($K = \Gamma$), constancy follows from continuity and the connectedness of X.

REMARK. Unfortunately the constancy proof in the case of finite K does not use the form of (6.11). Even with the continuity hypothesis we have been unsuccessful in using (6.11) to prove constancy.

LEMMA 6.13. *With notations as above assume $\Lambda(\cdot)$ is constant. Then Λ is a normal subgroup of K, and if $\Lambda \neq K$, K/Λ is isomorphic to X.*

PROOF. If $\Lambda(x) \equiv \Lambda$, then by (6.11) $\gamma_j \Lambda \gamma_j^{-1} = \Lambda$ for $1 \leqq j \leqq r$. Since Γ is dense in K, Λ must be normal. Let $\eta: K \to K/\Lambda$ be the canonical projection. Let $K_0 = K/\Lambda$.

Recall that $z_1(w) = w(0)$. Define $\zeta_1(w) = \eta z_1(w)$, and then define T_0: $W \times K_0 \to W \times K_0$ by $T_0(w, k_0) = (\sigma w, \zeta_1(w)k_0)$. If $\Lambda_0(w)$ is defined analogously to $\Lambda(w)$, then $\Lambda_0(w) = \{e\}$ for all w. (The map π: $W \times K \to W \times K_0$ given by $\pi(w, k) = (w, \eta k)$ satisfies $\pi T = T_0 \pi$, and thus $\Lambda_0(w) = \eta \Lambda(w) = \{e\}$).

The triviality of Λ_0 implies that for any fixed $w \in W$ there exists for each $w' \in W$ a unique element $\xi(w') \in K_0$ such that $(w', \xi(w')) \in \mathcal{O}((w, e))$. We have $\xi(\sigma w') = \zeta_1(w')\xi(w')$, $w' \in W$, and $\xi(\cdot)$ is continuous because $\mathcal{O}((w, e))$ is closed. Therefore, by an argument like the one used in Lemma 6.9, if $\tau w' = \tau w''$, then $\xi(w') = \xi(w'')$. Let $f_0(x) = \eta f(x)$, and define $\xi(x) = \xi(w_x)$. Then $\xi(\cdot)$ is continuous, and

$$\xi(x + \theta) = f_0(x)\xi(x). \tag{6.14}$$

In this case f_0 can have no discontinuity (because if $\delta_i = \eta\gamma_i$, (6.14) implies $\delta_i\xi(t_i) = \delta_{i+1}\xi(t_i)$, $0 \leqq i < r$, and $\delta_r\xi(0) = \delta_1\xi(0)$). Thus, $\eta\gamma_i = \eta\gamma_j = \delta$, all i, j, and (6.14) is

$$\xi(x + \theta) = \delta\xi(x). \tag{6.14'}$$

By (6.14′) the map $n\theta \to \delta^n$ extends to a continuous homomorphism $\phi: X \to K_0$. Since $\{\delta^n\}$ is dense in K_0, if $\Lambda \neq K$, then $\delta \neq 0$. It follows that $K_0 \approx X/F$, where F is a finite subgroup of X, and therefore $K_0 \approx X$.

If $\Gamma = K$ is finite, the alternative $K_0 \approx X$ is impossible, and we conclude $\Lambda = K = \Gamma$. That is, (T, Z) is minimal.

PROOF OF THEOREM 1.7. We will use the fact that (T, Z) is minimal under the hypotheses of Theorem 1.7. For each $\lambda \in \Gamma$ define Z_λ to be the set $Z_\lambda = \{(w, \lambda) \mid w \in W\}$. Z_λ is open in Z. By minimality there exists an integer $l = l_\lambda$

such that $Z = \bigcup_{j=1}^{l} T^{-j} Z_\lambda$. Let $L = \max_{\lambda \in \Gamma} l_\lambda$. If $x \in X$, $\lambda \in \Gamma$, there exists an integer $j, 1 \leqq j \leqq l_\lambda$, such that $T^j(w_x, e) \in Z_\lambda$. In particular, $z_j(w_x) = \lambda$. But $z_j(w_x) = f^{(j)}(x)$. The theorem is proved.

REMARK. If the answer to (6.12) were "yes", and if $\{\gamma_i \gamma_j^{-1} \mid 1 \leqq i, j \leqq r\}$ also generate a dense subgroup of K, then the alternative $K_0 \approx X$ in Lemma 6.13 is impossible because necessarily then δ in that lemma is $\delta = e$. Then (T, Z) is minimal. The replacement for Theorem 1.7 is: Let U be a neighborhood of e in K. There exists L such that for every $x \in X$ and $k \in K$ there is a j, $1 \leqq j \leqq L$, with $f^{(j)}(x) \in kU$.

REMARK. It was pointed out to us by N. Markley that minimality of (T, Z) under the hypotheses of Theorem 1.7 implies the minimality of (T, Z) for $Z = W \times K$, K compact totally disconnected, and Γ a dense subgroup of K. It is possible to extend similarly Theorems 1.2 and 1.8. The reason for this is that it can be shown using Furstenberg's principle [1] and arguments as in [6] that $(T, X \times K, \mu)$ is ergodic if and only if (T, Z) has a unique invariant probability measure (is "uniquely ergodic"). If K is totally disconnected, then K is an inverse limit of finite groups, say $K = \lim_\alpha^{-1} \Gamma_\alpha$ with $\pi_\alpha : K \to \Gamma_\alpha$ the associated homomorphisms. Correspondingly, (T, Z) is an inverse limit of flows built up using $f_\alpha : X \to \Gamma_\alpha$, where $f_\alpha(x) = \pi_\alpha f(x)$. The latter flows are uniquely ergodic under the hypotheses of either Theorem 1.2 or Theorem 1.8, because $(T, X \times \Gamma_\alpha, \mu_\alpha)$ is ergodic. It is easy to see that an inverse limit of uniquely ergodic flows is uniquely ergodic, and therefore (T, Z) is uniquely ergodic. Another application of the equivalence mentioned above implies $(T, X \times K, \mu)$ is ergodic.

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